A link polynomial via a vertex-edge-face state model

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Abstract

We construct a 2-variable link polynomial, called W_L, for classical links by considering simultaneously the Kauffman state models for the Alexander and for the Jones polynomials. We conjecture that this polynomial is the product of two 1-variable polynomials, one of which is the Alexander polynomial.

We refine W_L to an ordered set of 3-variable polynomials for those links in 3-space which contain a Hopf link as a sublink.
[1]

1 Introduction and results

We work in the smooth category. All 2- and 3-dimensional manifolds are oriented.

The present paper should have been written 20 years ago, when the state models for the Alexander and for the Jones polynomials were discovered.

Let us fix a coordinate system (x, y, z) in $\mathbb{R}^3$ and let $pr : (x, y, z) \rightarrow (x, y)$ be the standard projection. Let $S^3 = \mathbb{R}^3 \cup \infty$ and let $S^2 = (x, y) - plane \cup \infty$. Let $L \hookrightarrow \mathbb{R}^3$ be an oriented link. We represent links as usual by diagrams D with respect to pr (see e.g. [1]). Let $A = z - axes \cup \infty$. A is a *meridian* of D if there is a half-plane bounded by the z-axes which intersects D in exactly one point. If the intersection index at this point is $+1$ then A is called a
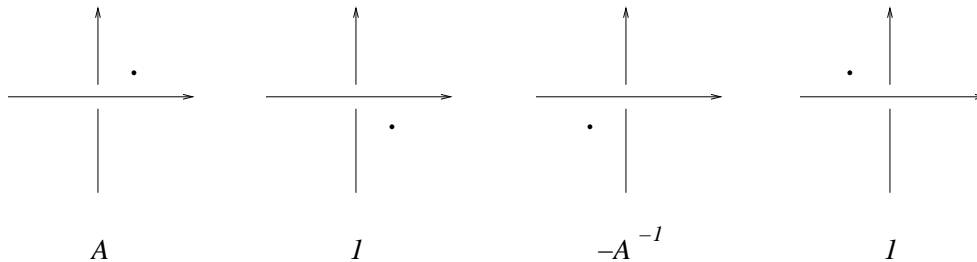

Figure 1:

positive meridian. Let D and D' represent the same knot in S^3 and such that A is a positive meridian for both of them. Then, as well known, D and D' represent the same knot in the solid torus $\mathbb{R}^3 \setminus z - axes$ (see e.g. [2]). For links we have to replace isotopy by isotopy which preserves the distinguished components (for which we have choosen the meridians).

More generally, in this paper we will study oriented links in the solid torus $V = \mathbb{R}^3 \setminus z - axes$.

Let D be an oriented link diagram. Its projection $pr(D)$ is an oriented graph in the annulus. We call the double points in $pr(D)$ the *vertices*, the arcs which connect the double points the *edges* and the components of its complement in the plane or in the annulus the *faces*.

Kauffman has constructed a vertex-face state model for the Alexander polynomial (see [6]) and a vertex-edge state model for the Jones polynomial (see [7]). For the convenience of the reader we remaind the definitions here.

Let D be an oriented and connected diagram of a link in $\mathbb{R}^3$. There are exactly two more regions in $\mathbb{R}^2 \setminus pr(D)$ than crossings of D. We mark two adjacent regions by stars (i.e. their boundaries in $\mathbb{R}^2$ have a common edge in the 4-valent graph $pr(D)$). A state T assignes now to each crossing of $pr(D)$ a dot in exactly one of the four local quadrants in $\mathbb{R}^2 \setminus pr(D)$ and such that in each region of $\mathbb{R}^2 \setminus pr(D)$, besides the regions marked by the stars , there is exactly one dot. To each quadrant we associate a monomial as shown in Fig. 1. Notice, that if we switch the crossing then the monomial is replaced by its inverse.

To each state T we associate now the product of the monomials corresponding to the dots. The *Alexander polynomial* $\Delta_L(t) \in \mathbb{Z}[t, t^{-1}]$ is then the sum of all products of monomials over all possible states T. Here , L denotes the link represented by D. Kauffman shows that $\Delta_L(t)$ is an isotopy invariant of L , that it does not depend on the choice of the adjacent regions
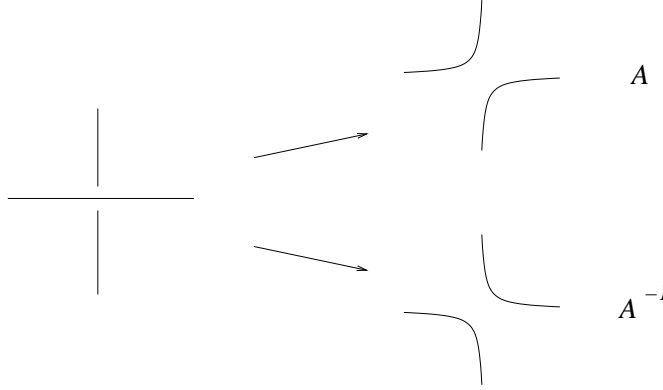
Figure 2:

with a star and that it actually coincides with the Alexander polynomial.

The *Kauffman bracket* in $\mathbb{R}^3$ is defined as follows: a state S splitts $pr(D)$ at each double point in exactly one of the two possible ways. To each such splitting we assigne a monomial as shown in Fig. 2.

Notice, that again if we switch the crossing then the monomial is replaced by its inverse. Let $/S/$ be the number of circles which are the result of splitting all double points of $pr(D)$ according to the state S. To each state S we associate the polynomial $< D, S >$ which is the product of all the monomials coming from the double points with $d^{/S/}$, where $d = -A^2 - A^{-2}$. The Kauffman bracket $< D >$ is then the sum of the polynomials $< D, S >$ over all states S. The Kauffman bracket is invariant under Reidemeister moves of type II and III but it is not invariant under Reidemeister moves of type I. But Kauffman shows that $(-A)^{-3w(D)} < D >$, where $w(D)$ is the *writhe* (compare e.g. [1]), is a link invariant which coincides (up to normalisation for the unknot) with the *Jones polynomial* $V_L(t)$ for $A = t^{-1/4}$.

In the case of the solid torus this state model can be refined (see [5]): let $/S/$ be now only the number of contractible circles in the annulus and let $[S]$ be the number of non contractible circles. We replace then $d^{/S/}$ by $d^{/S/} h^{[S]}$, where h is a new independent variable. For the Conway and the Kauffman skein modules of the solid torus see [12].

The starting point of the present paper is the following simple observation (already used in [4]): let D be a connected diagram in the solid torus $V = \mathbb{R}^3 \backslash z - axes$ and such that the corresponding link L is not contained in a 3-ball in the solid torus. (Evidently, we can always make D connected by performing
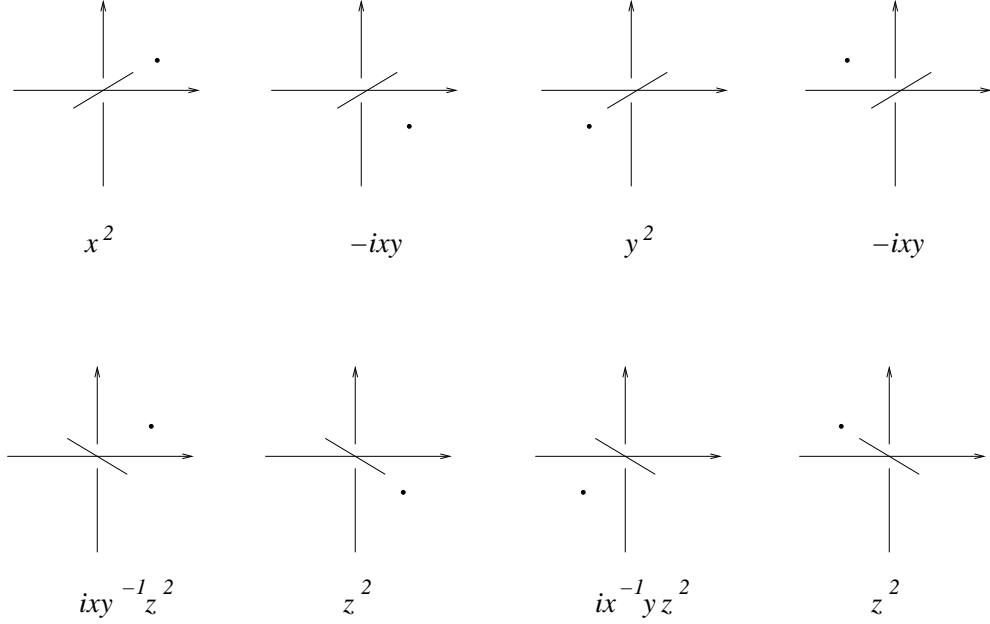
3

Figure 3:

just some Reidemeister II moves.) Then there are two *canonical regions* in $S^2 \setminus pr(D)$, namely those which contains ∞ and those which contains the origin $(0,0)$. We mark the canonical regions by the stars. Notice, that our star-regions are adjacent if and only if A is a meridian up to isotopy of the diagram D.

The important point is that we have no longer to prove invariance under the choice of the stars. This gives us the possibility to consider both types of state sums simultaneously.

Definition 1 *Let T be a Kauffman state for the Alexander polynomial. At each crossing we consider the two possible splittings (indicated by a small dash in the figures), i.e. the two Kauffman states S for the Jones polynomial. We associate to each positive crossing for each T and each S a (complex) monomial as shown in Fig. 3. Here, x y and z are independent variables.*

If we switch the crossing to a negative one then we associate for the same T and S the inverse monomial.

We set $d = ixyz^{-2} - ix^{-1}y^{-1}z^2$ and h is an independent variable as previously.

4

To each couple of states T and S, called a double state (T, S), *we associate the* double bracket $< D, T, S >$ *which is the product of the above defined monomials over all crossings of D and $d^{/S/}h^{[S]}$.*

We are now ready to define the polynomial invariant.

Definition 2 *Let D be an oriented diagram in the solid torus which represents a link L in the solid torus and let $w(D)$ be its writhe.*

The Laurent polynomial $W_L(x, y, z, h) \in \mathbb{Z}[i][x, x^{-1}, y, y^{-1}, z, z^{-1}, h]$ is defined as

$$W_L = (xyz^{-1})^{-2w(D)} \sum_T \sum_S < D, T, S >.$$

Here the sums are over all Kauffman states T for the Alexander polynomial and all Kauffman states S for the Jones polynomial.

It follows immediately from the definitions that $W_L(x, y, z, h)$ is homogenous in x, y, z of degree 0. Therefore, we can replace it by $W_L = W_L(x, y, 1, h)$ without loosing information.

Let D^2 be the unit disc in the (x, y)-plane and let V_1 be the solid torus $D^2 \times \mathbb{R} \setminus z - axes$. Let V_2 be the complementary solid torus $V \setminus V_1$.

Definition 3 *A link L in the solid torus V is called a* split link *if L is isotopic to a link $L_1 \cup L_2$ with the link L_1 contained in V_1 and L_2 contained in V_2. (If the axes A is a meridian of the link then our definition coincides with the usual definition of a split link in 3-space.)*

Let us consider oriented links in 3-space, i.e. links in the solid torus V such that A is a positive meridian. Then W_L is linear in h and we can forget the variable h. Let us consider in this case the homogenous Laurent polynomial $W_L(x, y, z)$ of degree 0.

The following theorem is our first result.

Theorem 1 *W_L is an isotopy invariant for oriented links in the solid torus. $W_L = 0$ for each split link L in the solid torus.*

Example 1 *Let K be the right-handed trefoil in 3-space. Then*

$$W_K = (x^4 + x^2y^2 + y^4)((xy)^{-10} + (xy)^{-8} + (xy)^{-4})$$

The polynomial for the left-handed trefoil $K!$ is obtained by replacing x and y by their inverses (which is not an immediate corollary of the definitions).

I am very grateful to Stepan Orevkov, who has written a computer program in order to calculate W_L (see [11]). Calculations with this program suggest the following conjecture.

Conjecture 1 *Let L be a link in 3-space. Then W_L is the product of a homogenous polynomial in x and y with a polynomial in xy. The homogenous polynomial coincides for $y = ix^{-1}$ with the Alexander polynomial $\Delta_L(x^4)$.*

Remark 1 *In order to find the polynomial W_L, we have of course associated to each of the eight pictures in Fig. 3 a new variable as well as to the corresponding pictures for a negative crossing together with the variable d. Invariance under the Reidemeister moves has led to a non-linear system of 17 variables and 60 equations. It turns out that this system has a unique solution which gives a homogenous Laurent polynomial of degree 0 of three variables .*

I am very grateful to Benjamin Audoux and to Delphine Boucher for their help in solving the above system.

Remark 2 *We have developed a machinery, called* one parameter knot theory*, which produces new knot polynomials from state sums in the solid torus of classical link polynomials (see [4] and also [3], which contains some necessary preparations). This has worked perfectly for the Kauffman state sums of the Alexander and of the Jones polynomials (see [4]).*

The present paper is a result of our search for new state sums. The new state sum takes into account not only the crossings and the arcs in the knot diagram which connect them, but also the components of its complement in the annulus.

State sums turned out to be very useful in order to categorify link polynomials in a combinatorial way (compare [9] and [10]).

Question 1 *Can W_L be categorified as a whole ?*

We refine now W_L with a new variable for a special class of links.

Let $L = L_1 \cup L_2 \cup L_3$ be an oriented link in 3-space such that $L_1 \cup L_2$ is a Hopf link and L_3 is an arbitrary link. We consider L up to isotopy which preserves this decomposition. Consequently, instead of L in S^3 we can equivalently study L_3 in the thickened torus $S^3 \setminus (L_1 \cup L_2) = T^2 \times \mathbb{R}$ (compare
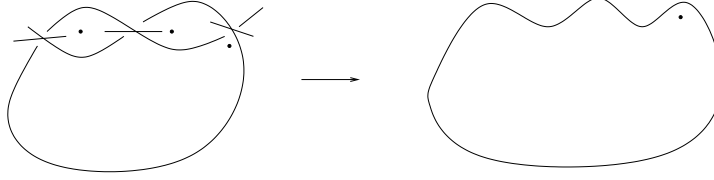
Figure 4:

e.g. [2]). Moreover, the natural projection of L_1 and L_2 into the 2-torus T^2 determines a distinguished pair of generators , say a and b, of $H_1(T^2)$.

Instead of the projection of L into the annulus we use now the projection of L_3 into the 2-torus in order to define $W(L_3 \hookrightarrow T^2 \times \mathbb{R})$ in exactly the same way as before besides the following two changings:

-there are no star regions, because the Euler characteristic of the torus vanishes (if there are no Kauffman states for the Alexander polynomial, then the invariant vanishes)

-the non contractible circles of a double state are all parallel in T^2 and hence represent the same homology class $|ma + nb|$. Here m and n are coprime integers and the homology class $ma + nb$ is only well defined up to sign. Hence, we have to replace the variable h by the homology class $|ma + nb|$. Contractible circles are traded to factors d as previously.

However, it turns out that $W(L_3 \hookrightarrow T^2 \times \mathbb{R})$ can be refined with a new variable, which comes from an unexpected relation between the Kauffman states for the Alexander polynomial and those for the Jones polynomial !

Definition 4 *A dot in a double state (T, S) is called* counting *if the spitting at the corresponding crossing is not in the direction of the dot. Each counting dot is* nearest *to a unique circle in the double state.*

For each circle in a double state we consider all its nearest counting dots on its left side and on its right side.

We show an example of a circle with exactly one nearest counting dot in Fig. 4.

Definition 5 *Let C be a circle in a double state (T, S). We chose its left side and its right side arbitrarily. Let v_+ be the number of nearest counting dots on its right side and let v_- be the number of nearest counting dots on its left side. The* weight *of the circle is then the natural number $v(C) = |v_+ - v_-|$.*
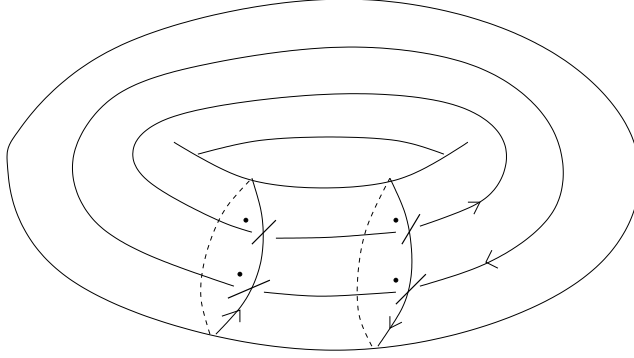
Figure 5:

Remark 3 *We could define the weight $v(C)$ in exactly the same way in our previous case of links in the solid torus. Unfortunately, it turns out that each contractible circle has always the weight 1 and each non contractible circle has always the weight 0.*

But in the case of the thickened torus the weight can be arbitrarily large for non contractible circles. We show an example in Fig. 5. The given double state has exactly two non contractible circles and each has the weight two.

Remark 4 *If we choose an orientation on C then the weight v could be defined as an integer, because the right side and left side are now defined canonically.*

We make the following observation (and we left the verification to the reader) : all non contractible circles in a double state have always the same weight and each contractible circle has the weight 1. Consequently, instead of a weighted configuration of non contractible circles all the information is already given by the following data: let C be a non contractible circle in the double state (T, S). Then we define

- the number of (parallel) non contractible circles , denoted by $s(C)$
- the (absolut value) of its homology class (with respect to the distinguished set of generators), denoted by $|C|$
- the weight of a non contractible circle, encoded by $t^{v(C)}$, where t is a new variable.

Definition 6 *Let D be the diagram in $T^2 \times \mathbb{R}$ represented by L_3. Then the refined double bracket $< D, T, S >_H$ is the product of the previously defined*

8
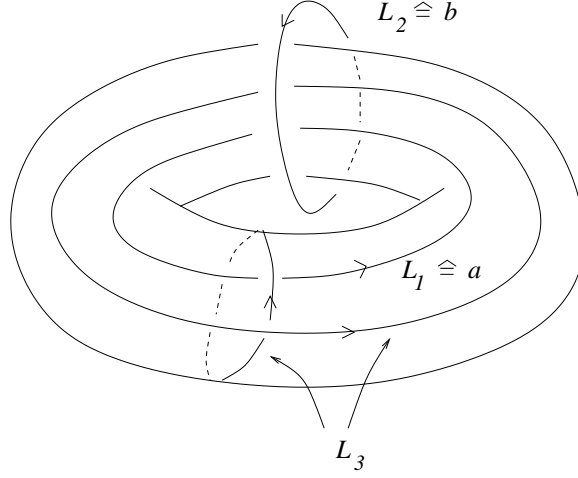
Figure 6:

monomials over all crossings of D and $d^{/S/}|C|^{s(C)}t^{v(C)}$. (Here, $|C|^{s(C)}$ is just the notation for $s(C)$ parallel curves in the given homology class (up to sign) $|C|$.)

Definition 7 *The* refined *polynomial* $W_L^H(x, y, z, t, |C|)$ *is defined as*
$\quad W_L^H = (xyz^{-1})^{-2w(D)} \sum_T \sum_S < D, T, S >_H$.
Here $< D, T, S >_H$ *is the refined double bracket.*

W_L^H can be seen as a 3-variable polynomial for each fixed element in $H_1(T^2)$.

Theorem 2 W_L^H *is an isotopy invariant for those oriented links in the 3-sphere which contain a distinguished Hopf link as a sublink.*

Example 2 *Let L be the link shown in Fig. 6. There is only one crossing and hence there are only eight double states. One easily calculates*
$\quad W_L^H = (x^2 + y^2 - 2ixyt)|a + b| + (2 + ixy^{-1}t + ix^{-1}yt)|a - b|$
(we have set $z = 1$).

Question 2 *Can W_L^H detect non invertibility of L ?*
Can it detect mutations which preserve the Hopf link $L_1 \cup L_2$?

Remark 5 W_L^H *could be easily generalized for links in handle bodies, but it would no longer lead to an invariant for classical links.*

9
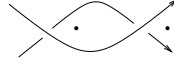
Figure 7:

Figure 8:

2 Proofs

We have to check that W_L is invariant under the Reidemeister moves (compare e.g. [1]).

Reidemeister I

The four Reidemeister I moves are shown in Fig. 7.

Notice, that the dot has to be in the newly created disc region, which can not be a star region. One easily calculates that the first two moves multiply $\sum_T \sum_S < D, T, S >$ by $x^2 y^2 z^{-2}$ and the other two moves multiply it by $x^{-2} y^{-2} z^2$ and, consequently, $W_L = (xyz^{-1})^{-2w(D)} \sum_T \sum_S < D, T, S >$ is invariant.

Reidemeister II

Let IIa be the move where the two tangencies at the autotangency point have the same orientation and let IIb be the move where they have opposite orientations. Let us consider IIa (the case IIb as well as the mirror images of these moves are completely analogous and are left to the reader). We splitt the set of all Kauffman states T for the Alexander polynomial into four subsets shown in Fig. 8 up to Fig. 11. We assume here that there are no stars in the regions shown in the figures. In each of the figures we consider now the four Kauffman states S for the Jones polynomial.

The results for two of the states are shown in Fig. 12 and in Fig. 13 (the remaining two states lead to identical pictures besides the dot, which has
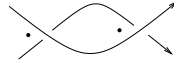
Figure 9:

Figure 10:

Figure 11:

slided in its region to the right, respectively downwards). We omit to write brackets and a dot which is not in a quadrant of a crossing means that there is a dot in the corresponding part of the region.

It follows that W_L is invariant under these Reidemeister II moves. This is still true if there are stars in the regions, because again there can not be a star in the newly created region bounded by the bigon. If the stars are situated as shown in Fig. 14 then it follows that $W_L = 0$. But notice that in this case L is necessarily a split link in the solid torus.

Reidemeister III

If one considers oriented diagrams then there are exactly eight different (local) types of Reidemeister III moves. Let us call the *positive Reidemeister III move* those in which all three involved crossings are positive. Fortunately, it turns out that in order to check that a polynomial is a knot invariant it suffices to check invariance only under the positive Reidemeister III move and under all types of Reidemeister II moves (see e.g. Section 1 in [2] and also Sections 2.3 and 2.4 in [4]).

Under a Reidemeister III move a triangle component of the complement of the diagram shrinks to a point and the link diagram has an ordinary triple point in the projection. There are three types of Kauffman states T for the Alexander polynomial at the triple point shown in Fig. 15.

For each of the three types we will consider just one of the six (respectively three) cases. The remaining cases are always analogue and are left to the reader. In the figures we draw only the planar image of the projection because

Figure 12:

11

$(\ ixy^{-1} - ixy^{-1}\ )$

$+ \ (\ x^2 z^{-2} - y^{-2} z^2 \ + \ ixy^{-1}d \ + \ z^2 y^{-2} - x^2 z^{-2} \ - \ ixy^{-1}d \ )$
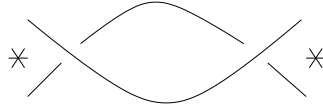
Figure 13:

Figure 14:

all crossings are positive . (The rest of the states T are of course identical outside of the corresponding figures.)

case IIIa

We show the corresponding states T before and after the move in Fig. 16.

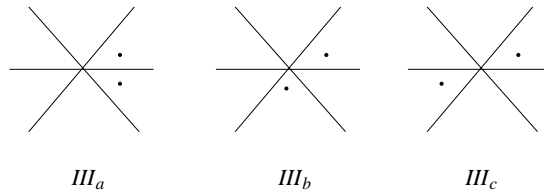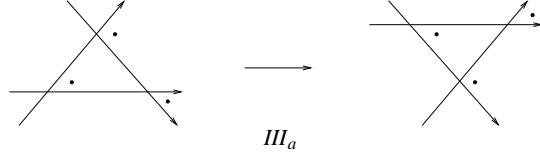Fig. 17 shows now the contribution to W_L before the move and Fig. 18 shows the contribution after the move.

case IIIb

We show the corresponding states T before and after the move in Fig. 19.

Fig. 20 shows the contribution to W_L before the move and Fig. 21 shows the contribution after the move.

case IIIc

We show the corresponding states T before and after the move in Fig. 22.

Fig. 23 shows the contribution to W_L before the move and Fig. 24 shows the contribution after the move.

III_a III_b III_c

Figure 15:

III_a

Figure 16:

$ix^5 y^{-1} z^2$ $+$ $(\, 2 \, ix^5 y^{-1} z^2 \, - \, ix^3 y^{-3} z^6$

$- x^4 y^{-2} z^4 d \,)$ $+$ x^6

$- x^4 y^{-2} z^4$ $- x^4 y^{-2} z^4$

Figure 17:

$(\, - x^4 y^{-2} z^4 d \; + \; 2 \, ix^5 y^{-1} z^2 \, - \, ix^3 y^{-3} z^6 \,)$

$+ \; ix^5 y^{-1} z^2$ $+$ x^6

$- x^4 y^{-2} z^4$ $- x^4 y^{-2} z^4$

Figure 18:

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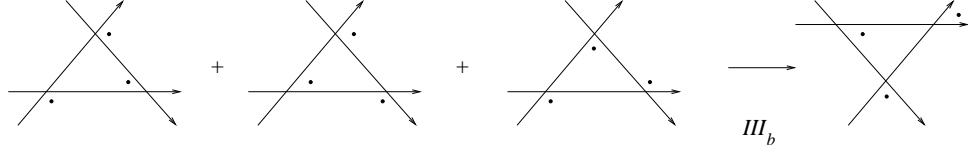
Figure 19:

$$(ixyz^4d \; + \; ix^3y^{-1}z^4d \; - \; ixyz^4d \; - z^6 \; - x^2y^{-2}z^6 \; + \; z^6$$

$$+ \; x^2y^2z^2 \; + \; x^4z^2 \; - x^2y^2z^2$$

$$+ \; x^2y^2z^2 \; + \; x^4z^2 \; - x^2y^2z^2 \;)$$

$$+ \; (\; x^2y^2z^2 \; + \; x^4z^2 \; - \; x^2y^2z^2 \;)$$

$$+ \; (\; ixyz^4 \; + \; ix^3y^{-1}z^4 \; - \; ixyz^4 \;)$$

$$+ \; (\; -ix^3y^3 \; - \; ix^5y \; + \; ix^3y^3 \;)$$

$$+ \; (\; ixyz^4 \; + \; ix^3y^{-1}z^4 \; - \; ixyz^4)$$

Figure 20:

14

$$x^4 z^2$$

$$+ \quad (\quad ix^3 y^{-1} z^4 d \quad + \quad 2\, x^4 z^2 \quad - \quad x^2 y^{-2} z^6 \quad)$$

$$+ \quad ix^3 y^{-1} z^4$$

$$- \quad ix^5 y$$

$$+ \quad ix^3 y^{-1} z^4$$

Figure 21:

Figure 22:

15

$$(\quad -y^2z^4d \quad + \quad y^2z^4d \quad - \quad ix^{-1}yz^6 \quad + \quad ix^{-1}yz^6$$

$$+ \quad 2\,ixy^3z^2 \quad - \quad 2\,ixy^3z^2 \quad)$$

$$+ \quad (\quad ixy^3z^2 \quad - \quad ixy^3z^2 \quad)$$

$$+ \quad (\quad -y^2z^4 \quad + \quad y^2z^4 \quad)$$

$$+ \quad (\quad x^2y^4 \quad - \quad x^2y^4 \quad)$$

$$+ \quad (\quad -y^2z^4 \quad + \quad y^2z^4 \quad)$$

Figure 23:

$$(\quad - \quad ix^3yz^2 \quad + \quad ix^3yz^2 \quad)$$

$$+ \quad (\quad x^2z^4d \quad - \quad x^2z^4d \quad - \quad 2ix^3yz^2 \quad + \quad 2ix^3yz^2$$

$$+ \quad ixy^{-1}z^6 \quad - \quad ixy^{-1}z^6 \quad)$$

$$+ \quad (\quad x^2z^4 \quad - \quad x^2z^4 \quad)$$

$$+ \quad (\quad -x^4y^2 \quad + \quad x^4y^2 \quad)$$

$$+ \quad (\quad x^2z^4 \quad - \quad x^2z^4 \quad)$$
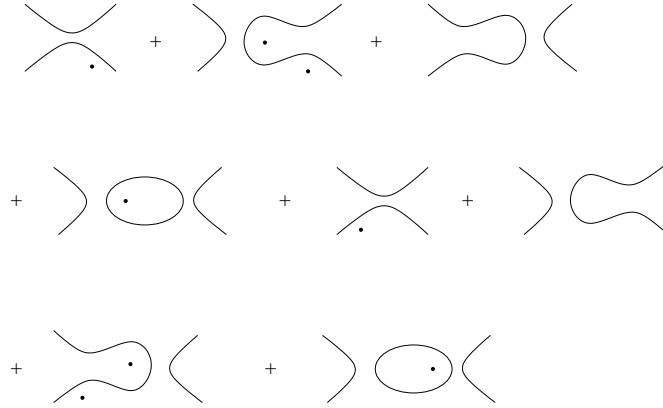
Figure 24:

Figure 25:

17

Figure 26:

It follows that W_L is invariant under these Reidemeister III moves. Again, stars in the figures would not change the above identities, because there can never be stars in the vanishing triangles.

Notice that the states of IIIc do not contribute at all to W_L. This was already observed for the Alexander polynomial in [4] , Remark 11.

Theorem 1 is proven.

The proof of Theorem 2 is completely analogous besides the following additional consideration: we have to prove that all corresponding circles of configurations which enter into the same equation have the same weight v. Notice that the weights are completely determined by the unoriented curves together with the dots in the torus T^2. Therefore, it suffices to consider unoriented immersed curves in T^2 instead of oriented link diagrams.

Reidemeister I

The new dot is either not a counting dot or it is a counting dot nearest to a contractible circle.

Reidemeister II

Fig. 8 (as well as Fig. 9) leads to the configurations shown in Fig. 25. We draw only the counting dots.

Fig. 10 (as well as Fig. 11) leads to the configurations shown in Fig. 26. It follows that all corresponding circles have the same weight.

Reidemeister III

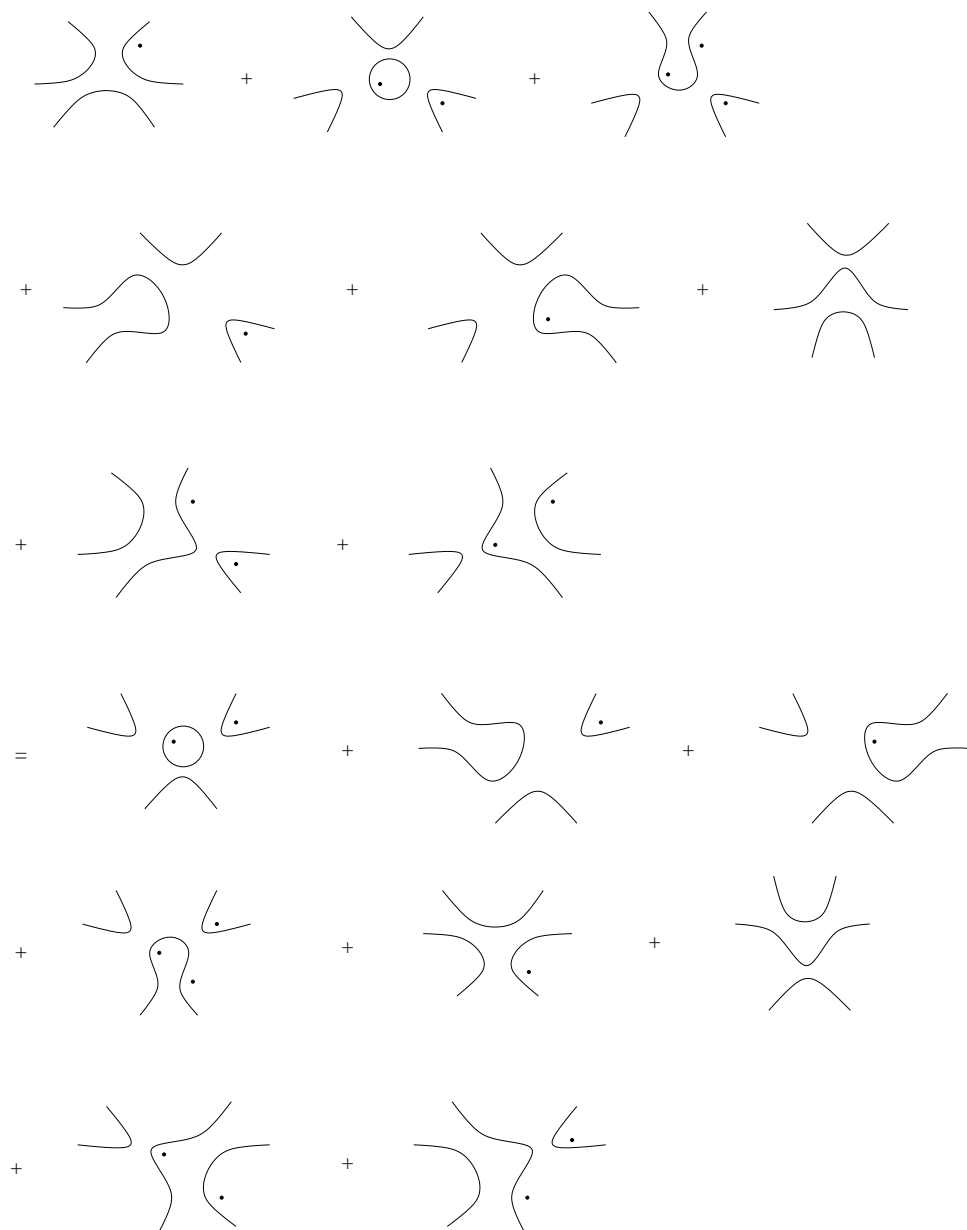We consider as an example the case IIIa. The other cases are analogous

Figure 27:

19

and we left the verification to the reader. Fig. 16 leads to the configurations shown in Fig. 27. It follows again that all corresponding circles have the same weight.

Theorem 2 is proven.

References

[1] Burde G., Zieschang H. : Knots, de Gruyter Studies in Mathematics 5, Berlin, (1985)

[2] Fiedler T. : Gauss Diagram Invariants for Knots and Links , Mathematics and Its Applications 532 , Kluwer Academic Publishers (2001)

[3] Fiedler T. , Kurlin V. : A one-parameter approach to knot theory, math. GT/0606381

[4] Fiedler T. : Knot polynomials via one parameter knot theory, math. GT/0612115

[5] Hoste J., Przytycki J. : An invariant of dichromatic links, Proc. Amer. Math. Soc. 105 (1989) 1003-1007

[6] Kauffman L. : Formal Knot Theory, Princeton University Press (1983)

[7] Kauffman L. : State models and the Jones polynomial, Topology 26 (1987) 395-407

[8] Kauffman L. : Knots and Physics, World Scientific, Singapur (1991)

[9] Khovanov M. : A categorification of the Jones polynomial, Duke Math. J. 101 (2000) 359-426

[10] Manolescu C., Ozsvath P., Sarkar S. : A combinatorial description of knot Floer homology, math. GT/0607691

[11] Orevkov S. : www.picard.ups-tlse.fr/ orevkov/

[12] Turaev V. : The Conway and Kauffman modules of the solid torus, J. Soviet. Math. 52:1 (1990) 2799-2805

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$$F_{\bigcirc}^{reg} \quad = \quad 1$$

$$F_{\diagdown\!\!\diagup}^{reg} \quad + \quad F_{\diagup\!\!\diagdown}^{reg} \quad = \quad m \quad (\quad F_{\smile}^{reg} \quad + \quad F_{)(}^{reg} \quad)$$

$$F_{\gamma}^{reg} \quad = \quad a \ F_{\smile}^{reg}$$

$$F_{\delta}^{reg} \quad = \quad a^{-1} \ F_{\smile}^{reg}$$

$$W_{\smile} \quad = \quad \overset{\bullet}{\diagup\diagdown} \quad + \quad \underset{\bullet}{\diagup\diagdown}$$

$$W_{\frown} \quad = \quad \overset{\bullet}{\diagdown\diagup} \quad + \quad \underset{\bullet}{\diagdown\diagup}$$

$$W_{\uparrow\!\!\rightarrow} \quad + \quad W_{-\uparrow\!\rightarrow} \quad =$$

$$(\ x^2\ +\ x^{-2})\ \overset{\bullet}{\diagup\diagdown} \quad + \quad (\ y^2\ +\ y^{-2}\ )\ \underset{\bullet}{\diagup\diagdown}$$

$$+ \quad i\ (\ x^{-1}y^{-1}\ -\ xy\ )(\ \diagup\diagdown \quad + \quad \diagup\diagdown\ )$$

$$+ \quad i\ (\ xy^{-1}z^2\ -\ x^{-1}yz^{-2}\ )\ (\ \diagdown\diagup \quad + \quad \diagdown\diagup\ )$$

$$+ \quad (\ z^2\ +\ z^{-2})(\ \underset{\bullet}{\diagdown\diagup} \quad + \quad \overset{\bullet}{\diagdown\diagup}\ )$$